HILBERT MODULES ASSOCIATED TO PARABOLICALLY INDUCED REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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Abstract. Given a measured space X with commuting actions of two groups G and H satisfying certain conditions, we construct a Hilbert $C^*(H)$-module $\mathcal{E}(X)$ equipped with a left action of $C^*(G)$, which generalises Rieffel's construction of inducing modules. Considering a semisimple Lie group G and a parabolic subgroup P with Levi component $L = MA$ and Langlands decomposition $P = MAN$, the Hilbert module associated to $X = G/N$ encodes the P-series representations of G coming from parabolic subgroups associated to P. We provide several descriptions of this Hilbert module, corresponding to the classical pictures of P-series. We then characterise the bounded operators on $\mathcal{E}(G/N)$ that commute to the left action of $C^*(G)$ as central multipliers of $C^*(L)$ and interpret this result as a globalised generic irreducibility theorem. Finally, we establish the convergence of intertwining integrals on a dense subset of $\mathcal{E}(G/N)$.

1. Introduction

In [17], M. A. Rieffel developed a general theory of induced representations for C^*-algebras by means of Hilbert modules. Namely, in the special case of group C^*-algebras, given a closed subgroup H of a locally compact group G, his construction yields a $C^*(H)$-Hilbert module E_H^G equipped with an action of $C^*(G)$ by bounded operators. This module *contains* all representations induced from H to G, in the sense that, for every representation $(\rho, \mathcal{H}_\rho)$ of H, there exists a map between $E_H^G \otimes_{C^*(H)} \mathcal{H}_\rho$ and the space of $\mathrm{Ind}_H^G \rho$, that preserves the scalar products on these Hilbert spaces, and intertwines the actions of $C^*(G)$. Another feature of [17] is the expression of Mackey's Imprimitivity Theorem through the Morita equivalence between $C^*(H)$ and $C_0(G/H) \rtimes G$.

P-series. In this work, we are interested in a particular type of induced representations of semisimple Lie groups. Let G be a linear connected semisimple Lie group with finite center, and P a cuspidal parabolic subgroup of G, with Langlands decomposition $P = MAN$. We denote by $\widehat{M}_d$ the discrete series of M and $\widehat{A}$ the unitary dual of A.

Definition 1. The *P-series representations* of G are the representations of the form

$$\mathrm{Ind}_P^G \sigma \otimes \chi \otimes 1,$$

where $\sigma \in \widehat{M}_d$ and $\chi \in \widehat{A}$.

Key words and phrases. Group C^*-algebras, Hilbert modules, induced representations, semisimple Lie groups, parabolic induction, principal series representations.

These representations play a key role in the Plancherel formula for G, established by Harish-Chandra in [7], that is in the description of the reduced dual $\widehat{G}_r$ as a measured space. Among the facts involved in that description, which will be explained in greater detail below, it is shown that P-series representations are generically irreducible [8], and that the families $\widehat{G}_P$ of classes of irreducible P-series representations are in fact parametrised by the conjugacy classes of Levi components $L = MA$ of the cuspidal parabolic subgroups P. In view of these facts, it appears that properly describing P-series representations within the Hilbert modules setting should involve modules over $C^*(L)$ rather than over $C^*(P)$, as Rieffel's theory permits. We will obtain such objects in Section 3, as a special case of the general construction of Section 2, dealing with group actions on a measured space satisfying certain hypotheses. Section 4 is devoted to providing different pictures of these modules, reflecting some classical porperties of P-series representations, while in Section 5 we prove a global analogue to the irreducibility theorem of Bruhat and Harish-Chandra by characterising some bounded invariant operators. Another application of this point of view is given in Section 6, where we prove the convergence of intertwining integrals.

As will be explained in Section 7, the modules discussed here also constitute the framework for a global theory of intertwining operators, developed in order to study the reduced dual as a non-commutative topological space, that is to analyse the reduced C^*-algebra $C^*_r(G)$ in terms of C^*-algebras associated to the Levi components of certain parabolic subgroups.

General notations and preliminaries. We refer the reader to [9, 10] and [15] for general facts about structure and representation theory of semisimple Lie groups, and to [13] for basic theory about Hilbert modules.

If X is topological space, we denote by $C_c(X)$ the space of compactly supported functions on X with complex values. If G is a locally compact group, we note dg a left Haar measure on G, and by Δ_G the corresponding modular function. The maximal and reduced C^*-algebras of the group are respectively denoted by $C^*(G)$ and $C^*_r(G)$. If A is a C^*-algebra and E, F are A-Hilbert modules, the set of bounded (adjointable) operators between E and F is denoted by $\mathcal{L}_A(E, F)$ or simply $\mathcal{L}(E, F)$, while the compact operators are denoted by $\mathcal{K}_A(E, F)$ or $\mathcal{K}(E, F)$. In the case $E = F$, the set $\mathcal{L}(E)$ is a C^*-algebra containing $\mathcal{K}(E)$ as a two-sided ideal.

We call *multipliers of* E the elements of $\mathcal{L}(A, E)$, also denoted $\mathcal{M}(E)$, where A is viewed as a Hilbert module over itself. The module E embeds in $\mathcal{M}(E)$ by associating to $\xi \in E$ the map $m_\xi : a \longmapsto \xi.a$ on A, with adjoint map $\langle \xi, \cdot \rangle$. If $E = A$, we recover one of the classical equivalent definitions of the *multiplier algebra* of A, and $\mathcal{M}(E)$ is a Hilbert module over $\mathcal{M}(A)$. The construction giving $\mathcal{M}(E)$ is functorial and for $T \in \mathcal{L}(E, F)$, we write $M(T) : \mathcal{M}(E) \to \mathcal{M}(F)$ the operator of left composition by T, with adjoint $M(T^*)$. For $\xi \in E$ and $\eta \in F$,

$$m_{\langle T.\xi, \eta \rangle} = \langle M(T).m_\xi, m_\eta \rangle$$

holds with inner products respectively taking values in F and $\mathcal{M}(F)$.

An example of elements in $\mathcal{M}(C^*(G))$ is given by extending left translations of compactly supported functions on G: for $g \in G$, we denote by U_g the multiplier of $C^*(G)$ defined by $U_g.f = f(g^{-1}\cdot)$ for any $f \in C_c(G)$.

The results presented here are part of the author's doctoral dissertation [5], written under patient and careful advisory of Pr. Pierre Julg in Orléans.

2. General construction

Let X be a locally compact topological space, G and H locally compact groups acting on X from the left and the right respectively. We assume that the actions commute and that the action of H is proper, hence implying local compactness of X/H. In what follows, we will also need a paracompactness assumption on X/H, which will always be satisfied in concrete examples, where X/H will turn out to be a compact manifold. Let us finally assume that X carries a G-invariant Borel measure μ, and that μ is H-relatively invariant with character δ_X, meaning that the relation $d\mu(g.x.h) = \delta_X(h)d\mu(x)$ is satisfied for any $g \in G$, $h \in H$. From the above data $G \curvearrowright (X, \mu) \curvearrowleft H$, we build a right $C^*(H)$-Hilbert module together with a left action of $C^*(G)$ by bounded operators. This module will be obtained as the completion of $C_c(X)$ with respect to an appropriate norm. Let us first describe the pre-Hilbert module structure on $C_c(X)$ over the dense involutive subalgebra $C_c(H)$ of $C^*(H)$. For any $f \in C_c(X)$ and $\varphi \in C_c(H)$, define

$$(f.\varphi)(x) = \int_H \frac{1}{\Delta_H(h)^{\frac{1}{2}}\delta_X(h)^{\frac{1}{2}}} f(x.h^{-1})\varphi(h)\, dh$$

for all $x \in X$.

Notation 1. The map $H \to \mathbb{R}_+^*$ defined by $\delta_X^{\frac{1}{2}}\Delta_H^{-\frac{1}{2}}$ will be denoted $\gamma_{X,H}$ or shortly γ_X when no confusion is likely to arise.

Proposition 1. *For $f, g \in C_c(X)$ and $h \in H$, let*

$$\langle f, g\rangle(h) = \gamma_{X,H}(h) \int_X \overline{f(x)} g(x.h)\, d\mu(x).$$

The map thus defined on $C_c(X) \times C_c(X)$ is a $C_c(H)$-valued inner product on $C_c(X)$.

Proof. Let $f, g \in C_c(X)$ and $\varphi \in C_c(H)$. The relations $\langle f, g\rangle^* = \langle g, f\rangle$ and $\langle f, g.\varphi\rangle = \langle f, g\rangle\varphi$ in $C^*(H)$ follow from straightforward calculations and the definition of $\gamma_{X,H}$. The positivity of $\langle f, f\rangle$ in $C^*(H)$ relies on the use of a *Bruhat section* for X over X/H: it is proved in [3] that the paracompactness of X/H guarantees the existence of a nonnegative bounded continuous function ψ on X such that $\operatorname{Supp}\psi \cap K.H$ is compact whenever K is a compact subset of X, and $\int_H \psi(xh)\, dh = 1$ holds for all $x \in X$. Let (ρ, V) be a unitary representation of H, the scalar product on V being denoted by $(\cdot,\cdot)$. The corresponding representation of $C^*(H)$ will still be denoted ρ. For $\xi, \eta \in V$,

$$\begin{aligned}
(\rho(\langle f, g\rangle)\xi, \eta) &= \int_H \gamma_X(h) \int_X \overline{f(x)} g(xh)\, (\rho(h)\xi, \eta)\; d\mu(x)\, dh \\
&= \int_H \gamma_X(h) \int_X \overline{f(x)} g(xh)\, (\rho(h)\xi, \eta) \int_H \psi(xh')\, dh'\, d\mu(x)\, dh \\
&= \int_X \psi(x) \int_{H\times H} \overline{f(xh'^{-1})} g(xh'^{-1}h)\, (\rho(h)\xi, \eta)\, \frac{\gamma_X(h)}{\delta_X(h')}\, dh'\, dh\, d\mu(x).
\end{aligned}$$

For $u \in C_c(H)$ and $x \in X$, let $\tilde{u}_x(h) = \gamma_X(h)u(x.h)$ for all $h \in H$. Then,

$$(\rho(\langle f,g\rangle)\xi,\eta) = \int_X \psi(x)\int_H \left(\tilde{f}_x^* * \tilde{g}_x\right)(h)\,(\rho(h)\xi,\eta)\;dh\,d\mu(x)$$

$$(\dagger) \qquad = \int_X \psi(x)\left(\rho(\tilde{f}_x^* * \tilde{g}_x)\xi,\eta\right)\,d\mu(x).$$

The above computations are justified by the fact that the support of

$$x \mapsto \int_H \left(\tilde{f}_x^* * \tilde{g}_x\right)(h)\,(\rho(h)\xi,\eta)\;dh$$

is contained in $\operatorname{Supp} f.H \cap \operatorname{Supp} g.H$ which has compact intersection with $\operatorname{Supp}\psi$. Setting $f = g$ in $(\dagger)$, it is clear that $\langle f,f\rangle$ is a positive element of $C_c(H) \subset C^*(H)$ and, taking ρ to be faithfull, that $\langle f,f\rangle = 0$ implies $f = 0$. □

Definition 2. For X, μ and H as above, let $\mathcal{E}_H(X,\mu)$ be the $C^*(H)$-Hilbert module obtained by completing $C_c(X)$ with respect to the norm $\|f\| = |\langle f,f\rangle|^{\frac{1}{2}}_{C^*(H)}$ and extending the action of $C_c(H)$ to $C^*(H)$.

Notation 2. When no confusion is likely to arise, this module will simply be denoted by $\mathcal{E}_H(X)$ or $\mathcal{E}(X)$.

Let us now describe the left action of $C^*(G)$.

Proposition 2. *Let X, μ, H and G be as above. There exists a $*$-morphism*

$$C^*(G) \longrightarrow \mathcal{L}_{C^*(H)}\left(\mathcal{E}(X)\right)$$

Proof. The action is first given at the level of the dense subalgebra $C_c(G)$. For $f \in C_c(X)$ and $\phi \in C_c(G)$, let

$$(\phi.f)(x) = \int_G \phi(g)f(g^{-1}x)\,dg$$

for all $x \in X$. Thus defined, $\phi.f$ belongs to $C_c(X)$ and Proposition 1 ensures that $\langle \phi.f,\phi.f\rangle$ is a positive element of $C^*(H)$. Let p be a state on $C^*(H)$, and ν_p the associated positive type Radon measure on H. Then the same computation as in the proof of Proposition 1 shows that

$$p\left(\langle f_1,f_2\rangle\right) = \int_H \langle f_1,f_2\rangle(h)\,d\nu_p(h) = \int_X \psi(x)\int_H \left(\tilde{f_1}_x^* * \tilde{f}_{2x}\right)(h)\,d\nu_p(h)\,d\mu(x),$$

for $f_1, f_2 \in C_c(X)$, using the same notations as above. It follows that the map $(f_1,f_2) \mapsto p\left(\langle f_1,f_2\rangle\right)$ provides an inner product on $C_c(X)$. Consider the Hilbert space obtained from $C_c(X)$ by completion with respect to the associated norm, denoted $\|\cdot\|_{(p)}$. Left translations yield a representation π_p of G on this space and the G-invariance of μ implies that π_p is unitary. Moreover, if $g \to g_0$ in G, then $\pi_p(g)f$ uniformly converges to $\pi_p(g_0)f$, while the supports remain in a fixed compact subset of X, so that π_p is strongly continuous. Still noting π_p for the integrated form of this representation, we obtain that

$$p\left(\langle \phi.f,\phi.f\rangle\right) = \|\pi_p(\phi.f)\|^2_{(p)} \le \|\phi\|^2_{C^*(G)}\,\|\pi_p(f)\|^2_{(p)} = \|\phi\|^2_{C^*(G)}\,p\left(\langle f,f\rangle\right).$$

Since this inequality holds for any state p of $C^*(H)$, it follows that

$$\langle \phi.f,\phi.f\rangle \le \|\phi\|^2_{C^*(G)}\,\langle f,f\rangle$$

in $C^*(H)$. Straightforward computations finally show that the left action of $C^*(G)$ commutes to the right action of $C^*(H)$, and $\langle \phi.f_1, f_2\rangle = \langle f_1, \phi^*.f_2\rangle$ so that $C^*(G)$ acts by adjointable operators and the map $C^*(G) \to \mathcal{L}_{C^*(H)}(\mathcal{E}(X))$ is a morphism of C^*-algebras. □

Example 1. If $X = G$, then by construction, the module $\mathcal{E}(G)$ is the induction module E_H^G introduced in [17] by Rieffel.

Example 2. If $H = \{1\}$, then $\mathcal{E}(X) \simeq L^2(X, \mu)$, with the regular representation of $C^*(G)$ coming from the action of G on X.

Example 3. If X is reduced to a point, then $\mathcal{E}(X)$ may be identified to $C^*(H)$ considered as a Hilbert module over itself, $C^*(G)$ acting trivially.

The last two examples are in fact extreme cases of the following result, which describes $\mathcal{E}(X)$ when X arises as the product of some topological space with the group acting on the right. This will also be the case in Section 4.2, when dealing with quotients of the open cell of some Bruhat decomposition.

Theorem 1. *Let B be a paracompact Hausdorff space with a Borel measure db and H a locally compact group. Consider $X = B \times H$ with the action of H given by right translations on itself and equipped with a measure of the form $d\mu(b, h) = \eta(h)\, db\, dh$ where η is a continuous morphism from H to $\mathbb{R}_+^*$. Then*

$$\mathcal{E}(X) \simeq L^2(B) \otimes C^*(H).$$

Proof. First notice that the particular form of the action of H on X implies that $\delta_X = \eta\Delta_H$, whence $\gamma_{X,H} = \sqrt{\eta}$. For $(f, g) \in C_c(B) \times C_c(H)$ and $(b, h) \in X$, let $P(f, g)(b, h) = \eta(h)^{-\frac{1}{2}} f(b)g(h)$. Then P factorises through the algebraic tensor product $C_c(B) \otimes C_c(H)$ and the existence of a continuous partition of unity for B implies that the range of P is uniformly dense in $C_c(X)$. Simple caculations show that $P(f \otimes g * \varphi) = P(f \otimes g).\varphi$ for any $\varphi \in C_c(H)$, proving $C_c(H)$-linearity for P, and finally that

$$\langle P(f \otimes g), P(f \otimes g)\rangle = \|f\|_2^2\, g^* g$$

in $C^*(H)$. We then conclude that P extends to an isometry between the Hilbert modules $L^2(B) \otimes C^*(H)$ and $\mathcal{E}(X)$ with dense range, which ends the proof. □

Remark 1. The previous proposition extends to the case where $B \times H$ is a subset of X such that $\mu(X \setminus B \times H) = 0$. In such a case, although the action of G on X might not restrict to an action on $B \times H$, the isometry between $\mathcal{E}(X)$ and $L^2(B) \otimes C^*(H)$ allows to define an action of $C^*(G)$ on the latter.

3. The Hilbert module $\mathcal{E}(G/N)$

3.1. **Setting and notations.** In what follows, G is a connected semi-simple Lie group with finite center. Let K be a maximal compact subgroup in G, with Lie algebra $\mathfrak{k}$. The Lie algebra $\mathfrak{g}$ of G admits Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ which determines a Cartan involution θ. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}_\mathfrak{p}$ be an Iwasawa decomposition of $\mathfrak{g}$ and $G = KA_\mathfrak{p}N_\mathfrak{p}$ the corresponding Iwasawa decomposition of G. Now let $M_\mathfrak{p}$ (resp. $M'_\mathfrak{p}$) be the centraliser (resp. the normaliser) of $A_\mathfrak{p}$ in K. Then the group $B = M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ will be called the *standard minimal parabolic subgroup* of G. A closed subgroup P of G which is the normaliser of its Lie algebra and contains a conjugate of B will be called a *parabolic subgroup*. Let N be the *unipotent radical*

of P, that is the maximal connected normal subgroup of P consisting of unipotent elements. Then P admits a closed reductive subgroup L such that $P = LN$, diffeomorphically. Let A be a maximal connected split abelian subgroup in the center of L, and $M = \bigcap_\chi \operatorname{Ker}|\chi|$, where χ runs over the continuous homomorphisms $\chi : L \to \mathbb{R}^*$. Then $L = MA$. The group L is called the *Levi component* of P and the decomposition $P = (M \times A) \ltimes N$ is called the *Langlands decomposition* of P. Respectively denoting $\mathfrak{a}$ and $\mathfrak{n}$ the lie algebras of A and N, it is possible to choose an ordering on the $\mathfrak{a}$-roots such that $\mathfrak{n}$ is the sum of the root-spaces associated to positive roots. Denoting $\mathfrak{g}_\lambda$ the root-space associated to a root λ, let ρ be the half sum of positive roots counted with multiplicity: $\rho = \frac{1}{2}\sum_{\lambda>0}\dim(\mathfrak{g}_\lambda)\lambda$. Since N is nilpotent and L is reductive, they are both unimodular and the Haar measure dp on $P = L \ltimes N$ decomposes into $dp = dl\,dn$. Finally, let $\bar{\mathfrak{n}}$ denote the image $\theta\mathfrak{n}$ of $\mathfrak{n}$ under the Cartan involution and $\bar{N}$ be the corresponding analytic subgroup.

3.2. **The module $\mathcal{E}(G/N)$.** The central object in this paper is the Hilbert module obtained by performing the construction of the previous section in the case of Lie groups G and L satifying the above assumptions, acting on $X = G/N$. As a consequence of Iwasawa decomposition, it is possible to write $G = KMAN$. Notice that $M \cap MAN = K \cap M$ is compact, and that the parts of the decomposition of an element in G relatively to KM, A and N are unique. Also using smoothness of the Iwasawa decomposition, it follows that the right action of L on G/N is free and proper. The coset space $(G/N)/L$ identifies to the flag variety G/P, hence is a compact manifold. Since N is nilpotent and G semisimple, these groups are both unimodular. It implies the existence of a unique up to normalisation G-invariant measure μ on G/N. Exploiting the choice of a maximal compact subgroup K of G and, once again, the decomposition $G = KP$ and the fact that N is normal in P, we may identify G/N to $K \times M \times A$ as a topological space, and μ to $e^{2\rho\log(a)}dk\,dm\,da$.

The last identification immediately implies L-relative invariance for μ and allows to compute $\delta_{G/N}$. We recover these facts below, without using any choice of a maximal compact subgroup. In order to do so, let us first recall some classical notation.

Notation 3. Following [3], if Γ is a topological group and $\sigma \in \operatorname{Aut}(\Gamma)$, the *modular function* of σ, denoted $\operatorname{mod}^\Gamma(\sigma)$, or $\operatorname{mod}(\sigma)$ when no confusion is likely to happen, is defined by the equality, $\int_\Gamma f \circ \sigma = \operatorname{mod}(\sigma)^{-1}\int_\Gamma f$, holding for any integrable function f. Now if Γ is a subgroup of a bigger group Γ' and $g \in \Gamma'$ normalises Γ, we write $c_g(\gamma) = g\gamma g^{-1}$ for every $\gamma \in \Gamma$. In the case of an inner automorphism c_γ of Γ, it follows from the definitions that $\operatorname{mod}^\Gamma(c_\gamma) = \Delta_\Gamma(\gamma^{-1})$.

Proposition 3. *The unique G-invariant measure μ on G/N is relatively invariant with respect to the right action of L and $\delta_{G/N}(l) = e^{2\rho\log(a)}$ for $l = ma \in L$.*

Proof. We first prove that $\delta_{G/N}(l) = \operatorname{mod}^N(c_l)$ for any $l = ma \in L$. For f integrable function and $\dot{g}$ the class in G/N of $g \in G$, define $F(\dot{g}) = \int_N f(gn)\,dn$ so that

our normalisations of Haar measures give $\int_G f = \int_{G/N} F\, d\mu$. Thus,

$$\begin{aligned}\delta_{G/N}(l)\int_G f(g)\,dg &= \int_{G/N} F(\dot{g}l^{-1})\,d\mu(\dot{g}) = \int_{G/N}\int_N f(gl^{-1}n)\,dn\,d\mu(\dot{g})\\ &= \operatorname{mod}(c_l)\int_{G/N}\int_N f(gnl^{-1})\,dn\,d\mu(\dot{g}) = \operatorname{mod}(c_l)\int_G f(gl)\,dg\\ &= \operatorname{mod}(c_l)\int_G f(g)\,dg,\end{aligned}$$

hence the first part of the result and the expected equality. The rest of the proof reduces to the classical Lie algebra computation leading to the modular functions of parabolic subgroups. Identifying N to its Lie algebra *via* the exponential map, we need to compute the jacobian determinant $|\det(\mathrm{Ad}(l)|_{\mathfrak{n}})|$. The properties of Langlands decomposition imply that M is the product of a closed subgroup of K, hence compact and a connected reductive group with compact center. It follows that $|\det(\mathrm{Ad}(m)|_{\mathfrak{n}})| = 1$ for any $m \in M$. Finally, an element $a \in A$ acts on the root space $\mathfrak{g}_\lambda$ by $e^{\lambda \log(a)}$, so that $|\det(\mathrm{Ad}(a)|_{\mathfrak{n}})| = e^{2\rho \log(a)}$, which concludes the proof. □

Let us now turn to the properties of $\mathcal{E}(G/N)$, starting with the fact that it is adapted to the description of the P-series representations.

3.3. **Specialisation.** As explained in Section 1, it is possible to recover the particular induced representations by tensoring Rieffel's module with the Hilbert space of the inducing representation. That specialisation procedure is still available using the $C^*(L)$-module $\mathcal{E}(G/N)$. The existence of the corresponding maps essentially relies on the following result, relating $\mathcal{E}(G/N)$ to Rieffel's module $\mathcal{E}(G)$.

Proposition 4. *There is an isometric isomorphism of $C^*(L)$-Hilbert modules*

$$\mathcal{E}_L(G/N) \simeq \mathcal{E}_P(G) \otimes C^*(L),$$

that intertwines the left actions of $C^*(G)$ on both sides.

Proof. Let M_N denote the averaging map defined by $M_N(f)(\dot{g}) = \int_N f(gn)\,dn$ for any function f in $C_c(G)$ and $\dot{g} \in G/N$ the class of $g \in G$. A similarly defined map on $C_c(P)$ extends to a surjection $\varepsilon_N : C^*(P) \twoheadrightarrow C^*(L)$. Let $\alpha \in C_c(P) \subset C^*(P)$, and $f \in C_c(G) \subset \mathcal{E}(G)$. Using the decomposition of measure $dp = dl\,dn$ it is easily seen that

$$(*) \qquad M_N(f.\alpha) = M_N(f).\varepsilon_N(\alpha)$$

in $\mathcal{E}(G/N)$. Now for $l = ma \in L$,

$$\gamma_{G,P}(l) = \sqrt{\frac{\Delta_G}{\Delta_P}}(l) = \Delta_P^{-\frac{1}{2}}(l) = e^{\rho\log(a)} = \delta_{G/N}^{\frac{1}{2}}(l) = \sqrt{\frac{\delta_{G/N}}{\Delta_L}}(l) = \gamma_{G/N,L}(l).$$

Then it follows from a straightforward computation that

$$(**) \qquad \langle M_N(f_1), M_N(f_2)\rangle_{\mathcal{E}(G/N)} = \varepsilon_N\left(\langle f_1, f_2\rangle_{\mathcal{E}(G)}\right)$$

in $C^*(L)$ for every $f_1, f_2 \in C_c(G)$. Recall that the $C^*(L)$-valued inner product on $\mathcal{E}(G) \otimes C^*(L)$ is defined on elementary tensors by

$$\langle f_1 \otimes \varphi_1, f_2 \otimes \varphi_2\rangle = \langle \varphi_1, \varepsilon_N\left(\langle f_1, f_2\rangle_{\mathcal{E}(G)}\right)\varphi_2\rangle_{C^*(L)} = \varphi_1^*\varepsilon_N\left(\langle f_1, f_2\rangle_{\mathcal{E}(G)}\right)\varphi_2.$$

Let A be defined on $C_c(G) \times C_c(L)$ by $A(f,\varphi) = M_N(f).\varphi$. Equality $(*)$ proves that A factorises through $C_c(G) \otimes C_c(L)$, while $(**)$ implies that

$$\langle A(f_1 \otimes \varphi_1), A(f_2 \otimes \varphi_2)\rangle_{\mathcal{E}(G/N)} = \langle f_1 \otimes \varphi_1, f_2 \otimes \varphi_2\rangle_{\mathcal{E}(G)\otimes C^*(L)}.$$

Since M_N is onto, it follows that A also has dense range and the above equality shows that it extends to the expected isometric isomorphism. □

Remark 2. The modules $\mathcal{E}(G/N)$ were introduced in [16] by F. Pierrot under the tensor product form of the above statement. The interest of the approach to $\mathcal{E}(G/N)$ as a special case of the general construction of Section 2 is so far two-fold: first, it leads to the convenient realisation of $\mathcal{E}(G/N)$ given in Theorem 3 through the use of the general result describing $\mathcal{E}(X)$ when $X = B \times H$ (Proposition 1). Another interest of seeing $\mathcal{E}(G/N)$ as a completion of $C_c(G/N)$ is that it makes it possible to directly define intertwining integrals similar to the ones considered by A. W. Knapp and E. M. Stein in [11, 12] without needing any of the meromorphic continuation argument used by these authors, as will be seen in Section 6.

Let us now establish the existence of specialisation maps.

Corollary 1 (Specialisation)**.** *For $(\sigma,\chi) \in \widehat{M} \times \widehat{A}$, let $\mathcal{H}_{\sigma\otimes\chi}$ be the Hilbert space of the representation $\sigma \otimes \chi \otimes 1$ of P, and $\mathcal{H}^P_{\sigma,\chi}$ the space of the induced representation*

$$\pi^P_{\sigma,\chi} = \mathrm{Ind}_P^G\, \sigma \otimes \chi \otimes 1.$$

There exists a map

$$q_{\sigma,\chi} : \mathcal{E}_L(G/N) \otimes_{C^*(L)} \mathcal{H}_{\sigma\otimes\chi} \longrightarrow \mathcal{H}^P_{\sigma,\chi}$$

unitarily intertwining the actions of $C^*(G)$ on these Hilbert spaces.

Proof. Associativity of the tensor product allows to reduce the proof to applying the similar result obtained by Rieffel in the case of classical induction Hilbert modules. Proposition 4 provides a unitary equivalence

$$\mathcal{E}_L(G/N) \otimes_{C^*(L)} \mathcal{H}_{\sigma\otimes\chi} \simeq \big(\mathcal{E}_P(G) \otimes_{C^*(P)} C^*(L)\big) \otimes \mathcal{H}_{\sigma\otimes\chi}.$$

The result follows from composing this isomorphism with the specialisation maps of [17] (Theorem 5.12, p.228). □

Remark 3. Considering the case of a cuspidal parabolic subgroup P and letting σ run over $\widehat{M_d}$ shows in particular that the module $\mathcal{E}(G/N)$ 'contains' all the P-series representations of G.

We now turn to other realisations of $\mathcal{E}(G/N)$, encoding some classical features of P-series representations.

4. Different pictures

Notations in this section are the same as in the previous one.

4.1. **Induced picture.** According to the traditional definition, essentially due to G. Mackey for locally compact groups, induced representations act on spaces of sections of equivariant fiber bundles. In particular, for $(\sigma,\chi) \in \widehat{M}_d \times \widehat{A}$, the P-series representation $\pi^P_{\sigma,\chi}$ acts on the space of L^2-sections of the fiber product $G\times_{\sigma\otimes\chi\otimes 1}\mathcal{H}_{\sigma\otimes\chi\otimes 1}$ over the flag manifold G/P. The trivial behaviour of the inducing parameter on N allows to consider sections of the G-equivariant bundle

$$\begin{array}{c} G/N \times_{\sigma\otimes\chi} \mathcal{H}_{\sigma\otimes\chi} \\ \downarrow \\ G/P \end{array}$$

as the space of $\pi^P_{\sigma,\chi}$. In order to recover that point of view within the global approach provided by the module $\mathcal{E}(G/N)$, it is tempting to realise it as a space of sections of the G-equivariant bundle

$$\begin{array}{c} G/N \times_L C^*(L) \\ \downarrow \\ G/P \end{array}$$

where $C^*(L)$ can be viewed as the collection of all the Hilbert spaces $\mathcal{H}_{\sigma\otimes\chi\otimes 1}$. This realisation will be called the *induced picture* of $\mathcal{E}(G/N)$.

Let us denote by $\mathcal{E}_i^o$ the space of compactly supported continuous functions $F : G/N \to C^*(L)$ subject to the relation

$$F(x.l) = e^{\rho\log(a)}U_{l^{-1}}.F(x)$$

for any $x \in G/N$ and $l = ma \in L$, on which $C^*(L)$ acts by right multiplication. Let ψ be a Bruhat section for the action of L on G/N, as defined in the proof of Proposition 1. The existence of ψ follows from the (para)compactness of the coset space $(G/N)/L \simeq G/P$. We then define a $C^*(L)$-valued inner product on $\mathcal{E}_i^o$ by defining

$$\langle F_1, F_2\rangle_\psi = \int_{G/N} F_1(x)^* F_2(x)\psi(x)\, d\mu(x)$$

for $F_1, F_2 \in \mathcal{E}_i^o$. The problem of the dependence on ψ is settled by the following result.

Lemma 1. *The map $\langle\cdot,\cdot\rangle_\psi$ defined above does not depend on the choice of the Bruhat section ψ.*

Proof. Let ψ_1 and ψ_2 be two Bruhat sections on G/N and $u = \psi_1 - \psi_2$. For $x \in G/N$ be represented by $kma \in KMA$, the relation satisfied by any functions $F_1, F_2 \in \mathcal{E}_i^o$ implies that $F_1(x)^*F_2(x) = e^{-2\rho\log(a)}F_1(k)^*F_2(k)$. It follows from Iwasawa decomposition of measure that

$$\begin{aligned} \int_{G/N} F_1(x)^* F_2(x)u(x)\, d\mu(x) &= \int_{K\times MA} F_1(kma)^* F_2(kma)e^{2\rho\log(a)}\, dk\, dm\, da \\ &= \int_K F_1(k)^* F_2(k)\, dk \int_L u(kl)\, dl\, dk \end{aligned}$$

The last quantity is zero since $\int_L u(kl)\, dl = 0$ for all $k \in K$ by definition of ψ_1 and ψ_2. It follows that $\langle F_1, F_2\rangle_{\psi_1} = \langle F_1, F_2\rangle_{\psi_2}$. □

Notation 4. As a consequence of the above lemma, we can denote without ambiguity $\langle\cdot,\cdot\rangle_i$ the sesquilinear form on $\mathcal{E}_i^o$, regardless of the Bruhat section used to construct it.

Definition 3 (Induced picture)**.** The space obtained by completing $\mathcal{E}_i^o$ with respect to the norm induced by $|\langle\cdot,\cdot\rangle_i|$ and denoted $\mathcal{E}_i$ is called the *induced picture of* $\mathcal{E}(G/N)$.

The $C^*(L)$-module $\mathcal{E}_i$ carries a left action of $C^*(G)$, defined by convolution at the level of compactly supported functions, in the same way as the one of Proposition 2.

Proposition 5. *The map $f \mapsto \tilde{f}$ defined on the dense subset $C_c(G/N)$ of $\mathcal{E}(G/N)$ by*

$$\tilde{f}(x)(l) = e^{\rho \log(a)} f(x.l)$$

for $x \in G/N$ and $l = ma \in L$, takes values in $\mathcal{E}_i^o$. It is $C_c(L)$-linear and preserves the $C^*(L)$-valued inner products.

Proof. Let $f \in C_c(G/N)$, $x \in G/N$ and $l_0 = m_0 a_0, l = ma \in L$. Then $\tilde{f}(x) \in C_c(L)$ and $\tilde{f}(xl)(l_0) = e^{\rho\log(a_0)} f(xll_0)$. Since

$$\left[U_{l^{-1}}\tilde{f}(x)\right](l_0) = \tilde{f}(x)(ll_0) = e^{\rho\log(aa_0)} f(xll_0),$$

the expected relation $\tilde{f}(xl) = e^{-\rho\log(a)}U_{l^{-1}}\tilde{f}(x)$ holds, implying that $\tilde{f} \in \mathcal{E}_i^o$. The $C_c(L)$-linearity follows from a straightforward computation. We prove the isometry property: for $f_1, f_2 \in C_c(G/N)$,

$$\begin{aligned}
\langle\widetilde{f_1},\widetilde{f_2}\rangle_i(l_0) &= e^{\rho\log(a_0)}\int_{G/N}\int_L \overline{\widetilde{f_1}(x)(l)}\,\widetilde{f_2}(x)(ll_0)\,dl\,\psi(x)\,d\mu(x)\\
&= e^{\rho\log(a_0)}\int_{G/N}\int_L e^{2\rho\log(a)}\overline{f_1(xl)}f_2(xll_0)\psi(x)\,dl\,d\mu(x)\\
&= e^{\rho\log(a_0)}\int_{G/N}\overline{f_1(x)}f_2(xl_0)\int_L \psi(xl^{-1})\,dl\,d\mu(x)\\
&= \langle f_1,f_2\rangle_{\mathcal{E}(G/N)}(l_0)\int_L \psi(xl)\,dl = \langle f_1,f_2\rangle_{\mathcal{E}(G/N)}(l_0)
\end{aligned}$$

where ψ is any Bruhat section on G/N. □

The map of the above statement has dense range in $\mathcal{E}_i$, as we see by considering $F \in \mathcal{E}_i^o$ such that $F(x) \in C_c(L)$ for $x \in G/N$: then $F = \tilde{u}$ where $u : x \mapsto F(x)(1)$. The properties of $C^*(L)$-sesquilinearity and positivity of $\langle\cdot,\cdot\rangle_i$ can be obtained as consequences of the previous proposition, and the following theorem holds as an immediate corollary.

Theorem 2. *There is an isometric isomorphism of $C^*(L)$-Hilbert modules*

$$\mathcal{E}(G/N) \simeq \mathcal{E}_i.$$

Moreover, the left action of $C^*(G)$ on $\mathcal{E}_i$ given by convolution coincides with the one obtained by transporting it from $\mathcal{E}(G/N)$ via this isomorphism.

Remark 4. It is clear from the definition and Iwasawa decomposition that functions in $\mathcal{E}_i^o$ are determined by their restriction to K. It makes it possible to obtain a *compact picture* of $\mathcal{E}(G/N)$ as the completion of a space of functions $K \to C^*(L)$.

4.2. **Open picture.** The classical *open* or *noncompact* picture of P-series representations (see [9]) presents the nice feature that it allows to realise all these representations on Hilbert spaces which do not depend on the representation σ of M in the inducing parameter. The main structural fact involved in this realisation is the following consequence of Bruhat decomposition of G (proved for instance in [10]):

Proposition 6 (Open Bruhat cell). *The set $\bar{N}MAN$ is open in G and its complement has Haar measure* 0.

This fact enables us to describe $\mathcal{E}(G/N)$ as the tensor product of a Hilbert space and the right-acting C^*-algebra, using Theorem 1:

Theorem 3 (Open picture). *There is an isometric isomorphism of $C^*(L)$-Hilbert modules*

$$\mathcal{E}(G/N) \simeq L^2(\bar{N}) \otimes C^*(L).$$

Proof. It is a straightforward consequence of Theorem 1 and Remark 1, applied to $B = \bar{N}$ and $H = L$, for $\bar{N}L$ has measure 0 in G/N. More precisely, a dense submodule of $\mathcal{E}(G/N)$ is obtained by considering functions of the form

$$F : \bar{n}ma \longmapsto e^{-\rho \log(a)} f(\bar{n})\varphi(ma),$$

where $f \in C_c(\bar{N})$ and $\varphi \in C_c(L)$. □

In the two following sections, we turn to applications of the point of view provided by the module $\mathcal{E}(G/N)$ on P-series. Section 5 is devoted to the description of bounded self-intertwiners of $\mathcal{E}(G/N)$, giving an irreducibility result and we establish in Section 6 the convergence of intertwining integrals on a dense subset of $\mathcal{E}(G/N)$.

5. The irreducibility theorem

5.1. **Groups of real rank 1.** From now on, we shall assume that the real rank of G, that is the dimension of the abelian part in the Iwasawa decomposition of G, is 1. As a consequence, proper parabolic subgroups of G are necessary minimal. With our previous notations, we thus consider $P = B$, with subgroups in its Langlands decomposition $M = M_{\mathfrak{p}}$ compact, $A = A_{\mathfrak{p}}$ isomorphic to $\mathbb{R}_+^*$ and $N = N_{\mathfrak{p}}$. Besides the discrete series, obtained by parabolic induction when G, considered as a parabolic subgroup of itself, is cuspidal, the only P-series are the B-series, also called *principal series.* We also denote:

- α the smallest restricted root of $(\mathfrak{g} : \mathfrak{a})$
- p and q the respective dimensions of $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$

It follows that $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ and $\rho = \frac{1}{2}(p + 2q)\alpha$. The automorphisms $(c_a\,,\, a \in A)$ of $\bar{N}$ are called *dilations.*

The Weyl group $W = N_K(A)/Z_K(A)$ contains exactly one non-trivial element, denoted w, and the Bruhat decomposition of G writes $G = PwP \sqcup P$, so that the complement of $\bar{N}MAN$ in G is the class wP of w in G/P.

Notation 5. Elements in the dense subset $(G/N) \setminus wL$ can be written according to $\bar{N}MA$ in a unique way. For such an element this decomposition will be denoted

$$g = \boldsymbol{\bar{n}}(g)\boldsymbol{m}(g)\boldsymbol{a}(g)n.$$

We also denote $\boldsymbol{l}(g) = \boldsymbol{m}(g)\boldsymbol{a}(g)$.

The technical aspects of what follows essentially consist of analysis on the space $\bar{N}$ on which A acts as a one-parameter group of dilations. The main tool to handle this situation is a particular function, taking into account the fact that an element in A acting on $\bar{N}$ identified to its Lie algebra $\bar{\mathfrak{n}}$, dilates vectors in different root spaces with according coefficients.

Definition 4. The *norm function on* $\bar{N}$ is the function defined on $\bar{N} \setminus \{1\}$ by

$$|\bar{n}| = e^{\rho \log \boldsymbol{a}(w^{-1}\bar{n})}.$$

Elementary facts about this function can be found in [11] and [5]. In particular, it will be useful to know that the norm function is C^∞ on $\bar{N}\setminus\{1\}$ and is continuously extended to $\bar{N}$ by setting $|1| = 0$. Moreover, $|\bar{n}^{-1}| = |\bar{n}|$ for any $\bar{n} \in \bar{N}$ and the measure $\dfrac{d\bar{n}}{|\bar{n}|}$ is invariant under dilations.

Another feature in real rank 1, is the possibility to completely describe the left action of $C^*(G)$ in the open picture of $\mathcal{E}(G/N)$. The action is initially defined by convolution at the level of $C_c(G/N)$ and carried to $L^2(\bar{N}) \otimes C^*(L)$ *via* the isomorphism of Theorem 3. One may also consider the corresponding action of G defined by translations on $C_c(G/N)$ and extended to $\mathcal{E}(G/N)$. The next proposition describe this action in the open picture. For $f \otimes \varphi \in L^2(\bar{N}) \otimes C^*(L)$ and $g \in G$, we write $g.f \otimes \varphi$ for the action of G imported from the one on $\mathcal{E}(G/N)$. With the real rank 1 assumption, Bruhat decomposition proves that it is enough to consider $g \in \bar{N}MA$ and $g = w$.

Proposition 7. *Let $f \otimes \varphi \in L^2(\bar{N}) \otimes C^*(L)$, $\bar{n}_0 \in \bar{N}$ and $l_0 = m_0 a_0 \in MA$. Then,*

- $\bar{n}_0.(f \otimes \varphi) = \lambda_{\bar{N}}(\bar{n}_0)(f) \otimes \varphi$
- $l_0.(f \otimes \varphi) = e^{\rho \log(a_0)} f \circ c_{l_0^{-1}} \otimes U_{l_0}.\varphi$
- *For any $\bar{\nu} \in \bar{N} \setminus \{1\}$ and $\lambda \in L$,*

$$w.(f \otimes \varphi) = \frac{1}{|\bar{\nu}|} f(\bar{\boldsymbol{n}}(w^{-1}\bar{\nu})) \left(U_{\boldsymbol{l}(w^{-1}\bar{\nu})}\varphi\right)(\lambda).$$

Proof. It directly follows from the form of the isomorphism in Proposition 3. $\square$

Let us now turn to the main result of this section.

5.2. Bounded self-intertwiners.

Definition 5. Let A be a C^*-algebra. An element M in the multiplier algebra $\mathcal{M}(A) = \mathcal{L}_A(A)$ is said to be *central* if it satisfies the relation

$$M(ab) = aM(b)$$

for any $a, b \in A$.

Remark 5. Using approximate units proves that the algebra of central multipliers of A coincides with the center of $\mathcal{M}(A)$.

Theorem 4. *The elements of $\mathcal{L}_{C^*(L)}(\mathcal{E}(G/N))$ which commute to the left action of $C^*(G)$ are exactly the central multipliers of $C^*(L)$.*

The method of the proof consists in associating to such an operator a bilinear form on a submodule of test functions and study the properties of the associate distributional kernel.

In what follows, all distributions take values in Banach spaces. General theory may be found in [4], as well as the next lemma, which characterises the distributions satisfying some invariance properties.

Proposition 8. *Let M be a differentiable manifold and Γ a Lie group with Haar measure $d\gamma$. Let E be a Banach space and T an E-valued distribution on $M \times \Gamma$. If T is invariant under the transformations $(m,\gamma) \mapsto (m, \gamma_0\gamma)$, then there exists an E-valued distribution S on M such that*

$$\langle T, \varphi\rangle = \int_\Gamma \langle S, \varphi_\gamma\rangle \, d\gamma$$

where $\varphi_\gamma : m \mapsto \varphi(m,\gamma)$ whenever φ is a test function on $M \times \Gamma$.

Remark 6. As a special case, it follows that, up to a scalar factor, the only left-invariant distribution on a Lie group is the Haar measure.

Proof of Theorem 4. The right action of $C^*(L)$ extends to one of $\mathcal{M}(C^*(L))$ on $\mathcal{E}(G/N)$ and if T_M denotes right multiplication by $M \in \mathcal{M}(C^*(L))$, the centrality condition for M implies $C^*(L)$-linearity for T_M. The fact that T_M commutes to the action of $C^*(G)$ is trivial, and boundedness follows from the existence of an adjoint map for T_M, namely T_{M^*}.

Conversely, let $T \in \mathcal{L}_{C^*(L)}(\mathcal{E}(G/N))$, satisfying the centrality condition. Following notations introduced in Section 1, $\mathcal{M}(\mathcal{E}(G/N))$ denotes the $\mathcal{M}(C^*(L))$-Hilbert module $\mathcal{L}(C^*(L), E)$.

We shall work in the open picture of Section 4.2. Since $\mathcal{M}(\mathcal{E}(G/N))$ contains $L^2(\bar{N}) \otimes \mathcal{M}(C^*(L))$, there is an injection $L^2(\bar{N}) \hookrightarrow \mathcal{M}(\mathcal{E}(G/N))$ through which $f \in L^2(\bar{N})$ is identified with the multiplier $f \otimes 1_{\mathcal{M}(C^*(L))}$, also denoted m_f, so that $m_{f\otimes a} = m_f m_a$, for any $a \in C^*(L)$, once more using notations of Section 1. It also follows that

$$M(T)(m_{f\otimes a}) = M(T)(m_f)m_a,$$

hence for $f_1 \otimes a_1, f_2 \otimes a_2 \in \mathcal{E}(G/N)$,

$$\langle M(T)(m_{f_1\otimes a_1}), m_{f_2\otimes a_2}\rangle = m_{a_1^*}\langle M(T)(m_{f_1}), m_{f_2}\rangle m_{a_2}.$$

Let us now consider the map $B_T : C_c(\bar{N}) \times C_c(\bar{N}) \to \mathcal{M}(C^*(L))$ defined by

$$B_T(f_1, f_2) = \langle M(T)(m_{\overline{f_1}}), m_{f_2}\rangle,$$

and prove that it is a Radon measure on $\bar{N} \times \bar{N}$. Let K be a compact subset in $\bar{N} \times \bar{N}$, and $f_1, f_2 \in C_c(\bar{N})$ such that $\operatorname{Supp} f_1 \times \operatorname{Supp} f_2 \subset K$. Recall that if E is a Hilbert module over a C^*-algebra A, the identity $|\langle \xi, \eta\rangle|_A \leq \|\xi\| \langle \eta, \eta\rangle^{\frac{1}{2}}$ holds for any $\xi, \eta \in E$, hence the following equality in $\mathcal{M}(C^*(L))$:

$$|B_T(f_1, f_2)| \leq \|M(T)(m_{f_1})\| . \left|m_{f_2}\right| .$$

It follows that $\|B_T(f_1, f_2)\| \leq \|T\|.\|f_1\|_2.\|f_2\|_2$, where $\|T\|$ denotes the operator norm of T, hence continuity of B_T with respect to the topology of uniform convergence on K. Consequently, B_T defines a distributional kernel k_T on $\bar{N} \times \bar{N}$, so that it writes

$$B_T(f_1, f_2) = \int_{\bar{N}\times\bar{N}} f_1(\bar{n}_1) f_2(\bar{n}_2)\, k_T(\bar{n}_1, \bar{n}_2)\, d\bar{n}_1\, d\bar{n}_2.$$

Since the left action of G on $\mathcal{E}(G/N)$ preserves the inner product, commutation of T to this action implies that $B_T(\bar{n}_0.f_1, \bar{n}_0.f_2) = B_T(f_1, f_2)$.

Applying the diffeomorphism $(\bar{n}_1, \bar{n}_2) \mapsto (\bar{n}_1^{-1}\bar{n}_2, \bar{n}_2)$ of $\bar{N} \times \bar{N}$ and using Proposition 8 and Remark 6 we see that k_T satisfies the equation

$$k_T(\bar{n}_1, \bar{n}_2) = k_T(1, \bar{n}_1^{-1}\bar{n}_2).$$

It follows that defining a distribution c_T on $\bar{N}$ by $c_T(\bar{n}) = k_T(1, \bar{n})$ implies that $k_T(\bar{n}_1, \bar{n}_2) = c_T(\bar{n}_1^{-1}\bar{n}_2)$. Invariance under the A action will allow us to characterise this distribution. Namely, the action of an element $l = ma \in L$ on an elementary tensor $f \otimes \varphi \in L^2(\bar{N}) \otimes C^*(L)$ is given by the formula $l.f \otimes \varphi = e^{\rho \log(a)} f \circ c_l \otimes U_l.\varphi$ of Proposition 7.

Commutation to the L action implies that $B_T(a.f_1, a.f_2) = B_T(f_1, f_2)$ for any $a \in A$. It follows that

$$\begin{aligned} e^{-2\rho\log(a)} \int_{\bar{N}\times\bar{N}} & f_1(c_a(\bar{n}_1)) f_2(c_a(\bar{n}_2))\, k_T(\bar{n}_1, \bar{n}_2)\, d\bar{n}_1\, d\bar{n}_2 \\ &= e^{2\rho\log(a)} \int_{\bar{N}\times\bar{N}} f_1(\bar{n}_1) f_2(\bar{n}_2)\, k_T(c_a^{-1}(\bar{n}_1), c_a^{-1}(\bar{n}_2))\, d\bar{n}_1\, d\bar{n}_2 \\ &= \int_{\bar{N}\times\bar{N}} f_1(\bar{n}_1) f_2(\bar{n}_2)\, k_T(\bar{n}_1, \bar{n}_2)\, d\bar{n}_1\, d\bar{n}_2, \end{aligned}$$

the first equality resulting of $\mathrm{mod}^{\bar{N}}(c_a) = e^{-2\rho\log(a)}$. As a consequence the distribution c_T satisfies the following invariance property: $c_T(c_a(\bar{n})) = e^{-2\rho\log(a)} c_T(\bar{n})$, that is, for any test function φ on $\bar{N}$,

$$(\ddagger) \qquad \langle c_T, \varphi \circ c_a \rangle = \langle c_T, \varphi \rangle.$$

Proposition 9. *If the real rank of G is 1, then the Radon measures on $\bar{N}$ satisfying relation ($\ddagger$) are multiples of the Dirac measure.*

Proof. Since $\bar{N}$ is a simply connected nilpotent Lie group, it may be identified *via* the exponential map to its Lie algebra $\bar{\mathfrak{n}} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \simeq \mathbb{R}^p \oplus \mathbb{R}^q$, in a way that preserves measures and allows to identify spaces of test functions and distributions. Under these identifications, the action by dilations of A on $\bar{N}$ is given on $\bar{\mathfrak{n}}$ in terms of α by $c_a(\bar{n}) \simeq (\alpha(a)u, \alpha(a)^2 v)$ for $a \in A$ and $\bar{n} \in \bar{N}$ identified to $(u, v) \in \mathbb{R}^p \oplus \mathbb{R}^q$. We denote by $a.(u, v)$ this last expression.

Denote $r(u, v) = \left(\|u\|^4 + \|v\|^2\right)^{\frac{1}{4}}$ for $(u, v) \in \mathbb{R}^p \oplus \mathbb{R}^q$ and let c_T^0 be the restriction of c_T to the open subset $\bar{N} \setminus \{1\}$. The function r is the Lie algebraic analogue of the norm function introduced above. For $t \in \mathbb{R}_+$, we denote the surface of equation $r(u, v) = t$. Then, for $a \in A \simeq \mathbb{R}_+^*$, we have $r(a.(u, v)) = a\, r(u, v)$, and the map

$$(u, v) \longmapsto (r(u, v), \frac{(u, v)}{r(u, v)})$$

is a diffeomorphism between $\bar{N} \setminus \{1\}$ and $\mathbb{R}_+^* \times S_1$. Fix $\psi_0 \in C_c^\infty(S_1)$. If φ is a test function on $\mathbb{R}_+^*$, then $\varphi \otimes \psi_0$ is in $C_c^\infty(\mathbb{R}_+^*) \otimes C_c^\infty(S_1) \subset C_c^\infty(\bar{N} \setminus \{1\})$ and the map $\varphi \mapsto \langle c_T^0, \varphi \otimes \psi_0 \rangle$ is a homogeneous distribution on $\mathbb{R}_+^*$. Proposition 8 implies that $\langle c_T^0, \varphi \otimes \psi_0 \rangle$ is of the form

$$c(\psi_0) \int_0^{+\infty} \varphi(r) \frac{dr}{r}.$$

Because of the integral term, c_T^0 cannot be the restriction of a Radon measure as can be seen by considering an appropriate sequence of test functions.

It follows that the support of c_T is reduced to $\{1\}$, so that c_T is a combination of derivatives in the sense of distributions of the Dirac measure. The homogeneity condition finally proves that $c_T = \delta_1.1_{\mathcal{M}(C^*(L))}$ up to a constant. □

According to the above result, there exists a multiplier $U \in \mathcal{M}(C^*(L))$ such that $c_T = U\delta_1$. It follows that $k_T(\bar{n}_1, \bar{n}_2) = U.\delta_1(\bar{n}_1^{-1}\bar{n}_2)$, hence the following form of the bilinear form:

$$B_T(f_1, f_2) = \langle f_1, f_2\rangle_{L^2}\, U$$

for $f_1, f_2 \in L^2(\bar{N})$.

As a consequence, for $a_1, a_2 \in C^*(L)$,

$$\begin{aligned} \langle M(T)(m_{f_1} \otimes m_{a_1}), m_{f_2} \otimes m_{a_2}\rangle &= \langle f_1, f_2\rangle_{L^2}\, m_{a_1}^*\, U\, m_{a_2} \\ &= \langle f_1 \otimes U^*(a_1), f_2 \otimes a_2\rangle \end{aligned}$$

in $\mathcal{M}(C^*(L))$, and $M(T)$ coincides with right composition by U^*. This mal is $C^*(L)$-linear only if U satisfies the centrality condition, which concludes the proof of Theorem 4.

□

Remark 7. In the special case of $q = 0$, the distribution c_T is homogeneous of degree $-p$. If $G = \mathrm{SL}_2(\mathbb{R})$ for instance, $p = 1$ and the problem reduces to the Euler equation on $\mathbb{R}$, the solutions of which are known to be combinations of the Dirac measure δ_0 and the *principal value* distribution $\mathrm{Vp}\left(\frac{1}{x}\right)$. The latter being of order 1, it is not the restriction of a Radon measure, which implies that $c_T = \delta_0$ up to a constant factor.

The above result should be seen as the analogue at the level of Hilbert modules of the generic irreducibility theorem due to Harish-Chandra in the general cuspidal case, and to Bruhat in the minimal one. Indeed, the self-intertwiners reduce to what plays the role of scalars, that is the multipliers satisfying the extra centrality condition needed to make homotheties morphisms in this noncommutative context.

6. Convergence of intertwining integrals

The construction of intertwining operators achieved in [11] to precisely detect reducibility phenomena in the cases that were not settled by Bruhat's theory relies on a remark that certain integral formulas formally satisfy the required intertwining relations, although they are given by non-locally integrable kernels. The constructions then roughly proceeds in two steps. The first one consists in allowing the parameter in $\widehat{A}$ seen as a subset of $\mathfrak{a}' \otimes \mathbb{C}$ to take non-purely imaginary values. The corresponding integrals are then convergent and the operators are defined as meromorphic extensions. The second step consists in composing these operators, which, using Bruhat theory, produces some meromorphic scalar functions. These functions are related to densities in the Plancherel formula and the study of their poles allows to decide of the reducibility of induced representations.

6.1. **Preliminaries.** The proof that the intertwining integrals are convergent on a dense subset of functions in $\mathcal{E}(G/N)$ will be carried out in the open picture. In order to obtain a convenient expression, we first need to establish some formulas, relating the action of G on the compact flag manifold G/P to the action on $\bar{N}$, by means of the decomposition of almost all of G according to $\bar{N}MAN$. The following

lemma should be seen as the expression of translations by G in G/P, composed with the 'stereographic projection' of G/P on the euclidean space $\bar{N}$[1].

Lemma 2. *Let $g \in G$, $\bar{\nu}_0 \in \bar{N}$ such that $g\bar{\nu}_0 \in \bar{N}MAN$ and $\bar{\nu} \in \bar{N} \setminus \{1\}$. Set $\mu = w^2 \in M$. Then,*

(i) $\bar{\boldsymbol{n}}(g^{-1}\bar{\boldsymbol{n}}(g\bar{\nu}_0)) = \bar{\nu}_0$
(ii) $\bar{\boldsymbol{n}}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = c_\mu(\bar{\nu})$
(iii) $\boldsymbol{m}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = \mu\, \boldsymbol{m}(w\bar{\nu})^{-1}$
(iv) $\boldsymbol{a}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = \boldsymbol{a}(w\bar{\nu})^{-1}$
(v) $\boldsymbol{l}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = \mu\, \boldsymbol{l}(w\bar{\nu})^{-1}$

Proof. Write $g\bar{\nu}_0$ with respect to $\bar{N}P$ as $g\bar{\nu}_0 = \bar{\boldsymbol{n}}(g\bar{\nu}_0)p$. Then

$$g^{-1}\bar{\boldsymbol{n}}(g\bar{\nu}_0) = g^{-1}\bar{\boldsymbol{n}}(g\bar{\nu}_0)pp^{-1} = g^{-1}g\bar{\nu}_0 p^{-1},$$

and (i) is a consequence of unicity in the decomposition according to $\bar{N}MAN$. (ii) follows from the remark that if $g \in G$ is such that $\bar{\boldsymbol{n}}(g)$ exists and $m \in M$, then $\bar{\boldsymbol{n}}(gm) = \bar{\boldsymbol{n}}(g)$. Indeed,

$$\bar{\boldsymbol{n}}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = \bar{\boldsymbol{n}}(w\bar{\boldsymbol{n}}(w^{-1}\mu\bar{\nu})) = \bar{\boldsymbol{n}}(w\bar{\boldsymbol{n}}(w^{-1}c_\mu(\bar{\nu}))),$$

hence the result. Finally, if $w\bar{\nu} = \bar{n}_0 p_0$ with $p_0 = m_0 a_0 n_0$ and $w\bar{n}_0 = \bar{n}_1 p_1$ with $p_1 = m_1 a_1 n_1$, then

$$\bar{n}_0 = w^{-1}\bar{n}_1 p_1 = w\mu^{-1}\bar{n}_1 p_1 = wc_{\mu^{-1}}(\bar{n}_1)\mu^{-1}p_1$$

hence

$$w\bar{\nu} = wc_{\mu^{-1}}(\bar{n}_1)\mu^{-1}p_1 p_0 = wc_{\mu^{-1}}(\bar{n}_1)\mu^{-1}m_1 m_0 a_1 a_0 n',$$

for some $n' \in N$. Since $\boldsymbol{m}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = m_1$ and $\boldsymbol{a}(w\bar{\boldsymbol{n}}(w\bar{\nu})) = a_1$, formulas (iii) and (iv) follow, thus implying (v). □

The next result makes the effect of the 'translation' by w explicit from the point of view of the measure on $\bar{N}$.

Lemma 3. *For $f \in L^2(\bar{N})$,*

$$\int_{\bar{N}} f(\bar{\boldsymbol{n}}(w\bar{\nu}))e^{-2\rho \log \boldsymbol{a}(w\bar{\nu})}\, d\bar{\nu} = \int_{\bar{N}} f(\bar{\nu})\, d\bar{\nu}.$$

Proof. Since $w \in K$, the map $L_w : f \longmapsto f(w\cdot)$ preserves the $C^*(L)$-norm on $\mathcal{E}(G/N) \simeq L^2(\bar{N}) \otimes C^*(L)$. Denote P the map of Theorem 3 defined on elementary tensors of $C_c(\bar{N}) \otimes C_c(L)$ by

$$P(f \otimes \varphi) : \bar{n}ma \longmapsto e^{-\rho \log a} f(\bar{n})\varphi(ma).$$

For any $x \in \bar{N}MA$ and $l_0 \in L$, it is clear that $\bar{\boldsymbol{n}}(xl_0) = \bar{\boldsymbol{n}}(x)$ and $\boldsymbol{l}(xl_0) = \boldsymbol{l}(x)l_0$. Consequently, if $\bar{n}ma \in \bar{N}MA$,

$$P(f \otimes \varphi)(w\bar{n}ma) = e^{-\rho \log a}e^{-\rho \log \boldsymbol{a}(w\bar{n})} f(\bar{\boldsymbol{n}}(w\bar{n}))\varphi(\boldsymbol{l}(w\bar{n})ma),$$

and L_w is given on $L^2(\bar{N}) \otimes C^*(L)$ by

$$L_w(f \otimes \varphi)(\bar{n}ma) = e^{-\rho \log \boldsymbol{a}(w\bar{n})} f(\bar{\boldsymbol{n}}(w\bar{n}))\lambda_L(\boldsymbol{l}(w\bar{n}))(\varphi)(ma).$$

Since the $C^*(L)$-norm is given in the open picture by $|f \otimes \varphi|^2 = \|f\|_2 \varphi^*\varphi$ and preserved by L_w, denoting $f_w(\bar{n}) = e^{-\rho \log \boldsymbol{a}(w\bar{n})} f(\bar{\boldsymbol{n}}(w\bar{n}))$, we get

$$\|f_w\|_2^2 = \|f\|_2^2,$$

[1] In the case of $G = \mathrm{SL}_2(\mathbb{R})$, the group $\bar{N}$ is a real line on which the flag manifold $G/P \simeq \mathbb{S}^1$ surjects.

since the action of $\lambda_L(\boldsymbol{l}(w\bar{n}))$ does not affect the norm. This proves the proposition for positive functions, and the result follows by linear combination. $\square$

Let us finally describe the effect of conjugating $\bar{\nu}$ by elements of L on the decomposition of $w^{-1}\bar{\nu}$ according to $\bar{N}MAN$.

Lemma 4. *Let $l_0 = m_0a_0 \in L$ and $\bar{\nu} \in \bar{N}$. Then,*
(i) $\boldsymbol{m}(w^{-1}c_{l_0}(\bar{\nu})) = c_w(m_0)\boldsymbol{m}(w^{-1}\bar{\nu})m_0^{-1}$
(ii) $\boldsymbol{a}(w^{-1}c_{l_0}(\bar{\nu})) = a_0^{-2}\boldsymbol{a}(w^{-1}\bar{\nu})$
(iii) $\boldsymbol{l}(w^{-1}c_{l_0}(\bar{\nu})) = c_w(m_0)\boldsymbol{l}(w^{-1}\bar{\nu})l_0^{-1}a_0^{-1}$

Proof. (iii) clearly follows from (i) and (ii). For those,

$$\begin{aligned} w^{-1}l_0^{-1}\bar{\nu}l_0 &= c_{w^{-1}}(l_0^{-1})w^{-1}\bar{\nu}l_0 = c_{w^{-1}}(l_0^{-1})\bar{n}'\boldsymbol{l}(w^{-1}\bar{\nu})n'l_0 \\ &= \bar{n}''c_{w^{-1}}(l_0^{-1})\boldsymbol{l}(w^{-1}\bar{\nu})l_0n'', \end{aligned}$$

with $n', n'' \in N$ and $\bar{n}', \bar{n}'' \in \bar{N}$. (i) follows from identifying the M components. Taking into account the fact that c_w acts on A as the inverse map $a \mapsto a^{-1}$, identifying the M components proves (ii). $\square$

6.2. **Proof of the convergence.** Let us now turn to standard intertwining integrals. As explained at the beginning of the previous paragraph, the theory of intertwining operators relies on the possibility to make sense of certain integral formula. In the classical context, it is meant to intertwine the representations $\pi_{\sigma,\chi}$ and $\pi_{w.\sigma,w.\chi}$, that is in particular to turn N-invariant functions into $\bar{N}$-invariant functions. It appears that the intertwining properties would be satisfied by the operator I_w given on a N-invariant function F by

$$I_wF(g) = \int_{\bar{N}} F(gw\bar{n})\,d\bar{n}$$

if it made sense.

We will prove the following statement:

Theorem 5. *The standard integral*

$$\int_{\bar{N}} F(gw\bar{n})\,d\bar{n}$$

defines a linear map $C_c(G/N) \to C(G/N)$.

The proof will be carried out in the open picture. To this purpose, we denote $\mathcal{I}_w$ the integral formula obtained by applying I_w to elementary tensors in the open picture of $\mathcal{E}(G/N)$. Namely, for $f \otimes \varphi \in C_c(\bar{N}) \otimes C_c(L)$ and $x_0 = \bar{n}_0m_0a_0$,

$$\mathcal{I}_w(f\otimes\varphi)(x_0) = e^{\rho\log(a_0)}\int_{\bar{N}} e^{-\rho\log\boldsymbol{a}(x_0w\bar{\nu})}f(\bar{\boldsymbol{n}}(x_0w\bar{\nu}))\varphi(\boldsymbol{l}(x_0w\bar{\nu}))\,d\bar{\nu}.$$

Notation 6. The automorphism of $C^*(L)$ induced by the conjugation c_w of L is denoted $a \mapsto a^w$.

Proposition 10. *Assuming that either side is defined, the equality*

$$\mathcal{I}_w(f\otimes\varphi)(x_0) = \int_{\bar{N}} f(\bar{n}_0\bar{\nu})\left[U_{\boldsymbol{l}(w^{-1}\bar{\nu})}\varphi\right]^w(l_0)\,\frac{d\bar{\nu}}{|\bar{\nu}|}$$

holds for $f\otimes\varphi \in C_c(\bar{N})\otimes C_c(L)$ and $x_0 = \bar{n}_0m_0a_0$.

Proof. The identities

$$\boldsymbol{m}(x_0 w\bar{\nu}) = m_0 \boldsymbol{m}(w\bar{\nu})$$
$$\boldsymbol{a}(x_0 w\bar{\nu}) = a_0 \boldsymbol{a}(w\bar{\nu})$$
$$\bar{\boldsymbol{n}}(x_0 w\bar{\nu}) = \bar{n}_0 c_{l_0}(\bar{\boldsymbol{n}}(w\bar{\nu})) = \bar{n}_0 l_0 \bar{\boldsymbol{n}}(w\bar{\nu}) l_0^{-1}$$

are clear. They imply that

$$\mathcal{I}_w(f\otimes\varphi)(x_0) = \int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(w\bar{\nu})} f(\bar{n}_0 c_{l_0}(\bar{\boldsymbol{n}}(w\bar{\nu})))\varphi(l_0 \boldsymbol{l}(w\bar{\nu}))\, d\bar{\nu}.$$

The change of variables $\bar{\nu} \leftrightarrow \bar{\boldsymbol{n}}(w\bar{\nu})$ leads *via* Lemma 3 to the expression

$$\int_{\bar{N}} e^{-\rho\log\left(\boldsymbol{a}(w\bar{\boldsymbol{n}}(w\bar{\nu}))\boldsymbol{a}(w\bar{\nu})^2\right)} f\left[\bar{n}_0 c_{l_0}(\bar{\boldsymbol{n}}(w\bar{\boldsymbol{n}}(w\bar{\nu})))\right]\varphi\left[l_0 \boldsymbol{l}(w\bar{\boldsymbol{n}}(w\bar{\nu}))\right]\, d\bar{\nu},$$

which simplifies to

$$\int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(w\bar{\nu})} f\left[\bar{n}_0 c_{l_0}(c_\mu(\bar{\nu}))\right]\varphi\left[l_0 \mu \boldsymbol{l}(w\bar{\nu})^{-1}\right]\, d\bar{\nu},$$

using Lemma 2.

We computed the modular function $\mathrm{mod}^{\bar{N}} c_l = e^{-2\rho\log a}$ for $l = ma \in L$ earlier. Besides, since $\mu = w^2 \in M$, it follows that $wc_{\mu^{-1}}(\bar{\nu}) = w\mu^{-1}\bar{\nu}\mu = w^{-1}\bar{\nu}$ for $\bar{\nu}\in\bar{N}$ hence $\boldsymbol{a}(wc_{\mu^{-1}}(\bar{\nu})) = \boldsymbol{a}(w^{-1}\bar{\nu})$ and $\boldsymbol{m}(wc_{\mu^{-1}}(\bar{\nu})) = \boldsymbol{m}(w^{-1}\bar{\nu})\mu$, which leads to $\boldsymbol{l}(wc_{\mu^{-1}}(\bar{\nu})) = \boldsymbol{l}(w^{-1}\bar{\nu})\mu$. We deduce from the previous remarks and Proposition 4 that:

$$\begin{aligned}
\mathcal{I}_w(f\otimes\varphi)(x_0) &= \int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(wc_{\mu^{-1}}(\bar{\nu}))} f\left[\bar{n}_0 c_{l_0}(\bar{\nu})\right]\varphi\left[l_0\mu \boldsymbol{l}(wc_\mu\mu^{-1}(\bar{\nu}))^{-1}\right]\, d\bar{\nu} \\
&= \int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(w^{-1}\bar{\nu})} f\left[\bar{n}_0 c_{l_0}(\bar{\nu})\right]\varphi\left[l_0 \boldsymbol{l}(w^{-1}\bar{\nu})^{-1}\right]\, d\bar{\nu} \\
&= \mathrm{mod}^{\bar{N}}(c_{l_0^{-1}}) \int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(w^{-1}c_{l_0^{-1}}(\bar{\nu}))} f(\bar{n}_0\bar{\nu})\varphi\left[l_0 \boldsymbol{l}(w^{-1}c_{l_0^{-1}}(\bar{\nu}))^{-1}\right]\, d\bar{\nu} \\
&= \frac{e^{-2\rho\log a_0}}{\mathrm{mod}^{\bar{N}}(c_{l_0})} \int_{\bar{N}} e^{-\rho\log \boldsymbol{a}(w^{-1}\bar{\nu})} f(\bar{n}_0\bar{\nu})\varphi\left[a_0^{-1}\boldsymbol{l}(w^{-1}\bar{\nu})^{-1} c_w^{-1}(m_0)\right]\, d\bar{\nu}
\end{aligned}$$

Since $c_w(a_0) = a_0^{-1}$, it follows that

$$\mathcal{I}_w(f\otimes\varphi)(x_0) = \int_{\bar{N}} f(\bar{n}_0\bar{\nu})\varphi\left[\boldsymbol{l}(w^{-1}\bar{n}^{-1}\bar{\nu})^{-1} c_{w^{-1}}(l_0)\right] e^{-\rho\log \boldsymbol{a}(w^{-1}\bar{\nu})}\, d\bar{\nu}$$

Using the notations $\varphi\mapsto\varphi^w$ for the action of w on $C^*(L)$ and $|\bar{n}|$ for the norm function introduced before, we finally get the expected expression. □

Let us now establish the convergence for compactly supported functions in the open picture.

Proposition 11. *Let $f\otimes\varphi \in C_c(\bar{N})\otimes C_c(L)$ and $x_0 = \bar{n}_0 l_0 = \bar{n}_0 m_0 a_0$. The integral*

$$\mathcal{I}_w(f\otimes\varphi)(x_0) = \int_{\bar{N}} f(\bar{n}_0\bar{\nu})\left[U_{\boldsymbol{l}(w^{-1}\bar{\nu})}\varphi\right]^w(l_0)\,\frac{d\bar{\nu}}{|\bar{\nu}|}$$

is convergent.

Proof. Since $\operatorname{Supp} f$ is compact, the integration domain is a compact subset of $\bar{N}$. Therefore, f and φ being continuous, it is enough to study integrability near 1, where $\frac{1}{|\bar{n}|}$ may introduce singularity. Since l_0 is fixed, $\boldsymbol{l}(w^{-1}\bar{\nu})^{-1}l_0^{-1}$ runs out of any compact of L as $\bar{\nu} \to 1$. Indeed, identifying A to $\mathbb{R}_+^*$ *via* $a \mapsto e^{\rho \log(a)}$, it appears that $\boldsymbol{a}(w^{-1}\bar{\nu}) \to 0$ as $\bar{\nu} \to 1$. It follows that $\left[U_{\boldsymbol{l}(w^{-1}\bar{\nu})}\varphi\right]^w(l_0) = \varphi\left(\boldsymbol{l}(w^{-1}\bar{\nu})^{-1}c_w(l_0)\right)$ is 0 in a neighbourhood of $\bar{\nu} = 1$, so that the integral is well-defined. □

Let us now give the proof of Theorem 5. We want to prove that the standard intertwining integral

$$I_w F(x) = \int_{\bar{N}} F(xw\bar{n})\, d\bar{n}$$

makes sense for $F \in C_c(G/N) \subset \mathcal{E}(G/N)$.

Proof of Theorem 5. Let $F \in C_c(G/N)$. Since G has real rank 1, the Bruhat decomposition implies that $G = \bar{N}P \sqcup w^{-1}P$, as we have seen before. Denote $p : G/N \twoheadrightarrow G/P$ the natural projection. Let us first assume that the projection $\operatorname{Supp} F$ does not contain the class $\left[w^{-1}\right] = [w]$ of w^{-1}. Then F is supported on a compact subset of $\bar{N} \times L$ and may thus be seen as a function on $\bar{N}$, taking values in $C_c(L)$.

Then the expression of $I_w F$ is identical to the one obtained in the open picture, up to variable separation :

$$I_w F(\bar{n}_0) = \int_{\bar{N}} U_{\boldsymbol{l}(w^{-1}\bar{n})} F(\bar{n}_0\bar{n})\, \frac{d\bar{n}}{|\bar{n}|}.$$

The proof of Proposition 11 applies, replacing the factors f and ϕ of an elementary tensor by the partial maps obtained by fixing separately the components relative to $\bar{N}$ and L.

We can now turn to the general case: let u be a compactly supported continuous function on G/P, taking values 1 on a neighbourhood of $[w]$ and 0 on a neighbourhood of $[1]$, so that $(u, 1-u)$ is a partition of unity. This partition can be lifted to G/N *via* p so that F decomposes to

$$F = p^* uF + p^*(1-u)F.$$

The case of $p^*(1-u)F$ was treated above: the support of this function avoids $\left[w^{-1}\right]$ by construction. Consider p^*uF: after composing with the left translation $L(w)$ the same condition on the support holds. Consequently, it makes sense to apply I_w to $L(w)p^*(1-u)F$. Since I_w is G-equivariant, it follows that it also applies to the second term in the sum, hence to F, which concludes the proof. □

Remark 8. The standard intertwining integral can also be expressed in the induced picture. More precisely, we proved in [5] that if $F \in \mathcal{E}_i^o$, this integral can be written

$$I_w F(x) = \int_{\bar{N}} U_{\boldsymbol{l}(w^{-1}\bar{n})} F(x\bar{n})\, \frac{d\bar{n}}{|\bar{n}|}$$

for $x \in G/N$, and is well defined because of Theorem 5 and the equivalence of the different pictures. This expression coincides with the one obtained for a function $F \in C_c(G/N)$ satisfying the appropriate condition of support, as used in the proof of Theorem 5, when we restrict it to $\bar{N}$.

To conclude this section, we remark that the interesting feature of this Hilbert module point of view concerning intertwining integrals is that no meromorphic argument is required to obtain a well-defined object. The drawback is that I_w does not take its values in $\mathcal{E}(G/N)$, as may be easily observed in the case of $\mathrm{SL}_2(\mathbb{R})$. This phenomenon is the analogue of the existence of poles in the theory of Knapp and Stein, and is measured by a certain Radon measure on L (see [5] for these two last facts). The main problem implied by this fact, is that it makes it impossible to compose the intertwining operators, which is a critical step in the method of [11, 12]. We came over this problem in [5] by other means in some special cases which will be the subject of another work.

7. Images of $C_r^*(G)$

Results of Lipsman [14] and Harish-Chandra [6] establish that the unitary equivalence classes of the P-series representations, for all possible P, can be partitioned according to the associativity classes of the inducing subgroups. This leads to the following decomposition of the reduced dual into a disjoint union

$$\widehat{G}_r = \bigsqcup_{[P],\, P\,\text{cuspidal}} \widehat{G}_P,$$

where $\widehat{G}_P$ denotes the set of irreducible components obtained from the P-series representations. This result, in relation with the Plancherel formula, discribes the reduced dual as a measured space. In order to understand it as a noncommutative topological space, we need to take the reducibility phenomena into account. Those were dealt with by Knapp and Stein [11, 12] by means of their intertwining operators, describing the intricate action of Weyl groups. The structure of the C^*-algebra was described in [18] for the cases where $\widehat{G}_r$ is Hausdorff, and in [19] by means of the operators of Knapp and Stein.

We indicate here how the Hilbert modules $\mathcal{E}(G/N)$ are likely to provide a more C^*-algebraic framework in order to analyse $C_r^*(G)$ with respect to the Levi components of cuspidal parabolic subgroups.

We no longer assume the real rank of G to be 1, but we consider $P = MAN$ minimal parabolic, which is a necessary and sufficient condition for M to be compact.

Proposition 12. *The left action of $C^*(G)$ on $\mathcal{E}(G/N)$ induces a $*$-morphism*

$$C_r^*(G) \longrightarrow C_0(\widehat{M} \times \widehat{A}, \mathcal{K})$$

Proof. Still denoting $\varepsilon_N : C^*(P) \twoheadrightarrow C^*(L)$ the canonical surjection, let τ be the map

$$\mathbb{C} \rtimes G \simeq C^*(G) \longrightarrow C(G/P) \rtimes G \simeq \mathcal{K}(\mathcal{E}(G))$$

coming from the Imprimitivity Theorem of Rieffel, taking into account the compactness of G/P. Let $\tau_N = \tau \otimes_{\varepsilon_N} 1$. Then

$$\tau_N : C^*(G) \longrightarrow \mathcal{K}(\mathcal{E}(G/N)).$$

Since P is amenable, all its actions are. The Imprimitivity Theorem implies that the action of G on G/P is Morita-equivalent to the one of P on G/G [1], hence amenable too [2]. It follows that $C(G/P) \rtimes G$ is isomorphic to $C(G/P) \rtimes_r G$, where $C_r^*(G)$ acts naturally. The situation is summed up in the following diagramm.

$$\tau_N : \begin{tikzcd} C^*(G) \arrow[r] \arrow[d, "\lambda_G"'] & C(G/P) \rtimes G \arrow[r] \arrow[d, "\simeq"'] & \mathcal{K}(\mathcal{E}(G/N)) \arrow[d, equal] \\ C_r^*(G) \arrow[r] \arrow[rr, dashed, bend left] & C(G/P) \rtimes_r G & C_0(\widehat{M} \times \widehat{A}, \mathcal{K}) \end{tikzcd}$$

□

It seems that the result extends to the cuspidal case with a $*$-morphism

$$C_r^*(G) \longrightarrow C_0(\widehat{M}_d \times \widehat{A}, \mathcal{K}).$$

The purpose of our further work will be to describe the image of this morphism, which will involve a significant use of the unitary operators that arise by normalising the intertwining integrals studied above (see [5] for the cases of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$).

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